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The Spectrum for Three-Times Repeated Blocks in a $S_3(2, 3, v)$

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Denote $R(v)$ be the set of k such that there is a $S_3(2, 3, v)$ having exactly k three-times repeated blocks. We prove that: if $v \equiv 1$ or 3 (mod 6) then $R(v) = \{0, 1, ..., t_v = v(v-1)/6\} - \{t_v - 1, t_v - 2, t_v - 3, t_v - 5\}$, if $v \equiv 5$ (mod 6) and $v \geqslant 17$ then $R(v) = \{0, 1, ..., s_v = [v(v-1) - 20]/6\} - \{s_v - 1, s_v - 2\}$, $R(11) = \{0, 1, ..., 15\} - \{11, 13, 14\}$, $R(5) = \{0\}$.

1. INTRODUCTION

Let $S_3(2, 3, v)$ denote the design (V, B), where V is a v-set and B is a collection of 3-subsets of V, called blocks, such that each 2-subset of V is contained in exactly three blocks. It is well known that a $S_3(2, 3, v)$ exists if and only if $v \equiv 1$ (mod 2). It is possible for B to contain three blocks, say b_1, b_2, and b_3, such that $b_1 = b_2 = b_3$; in this case b_1 is said to be a three-times repeated block. Let B_3 denote the set of three-times repeated blocks of B. In this paper we solve the following question: Given $v \equiv 1$ (mod 2) and a non-negative integer k, does there exist a $S_3(2, 3, v)$ (V, B) such that $|B_3| = k$?

A. Rosa and D. Hoffman have solved in [1] a similar (but not equivalent) problem for twofold triple systems.

2. PRELIMINARIES AND THE CASE $v \equiv 1$ OR 3 (mod 6)

Let (V, B) be a $S_3(2, 3, v)$. Simple counting arguments show that $|B| = v(v-1)/2$ and every element belongs to $3(v-1)/2$ blocks. For $v \equiv 1$ (mod 2), denote $R(v) = \{k/\exists(V, B)$ such that $|B_3| = k\}$. It is easy to show that $|B - B_3| > 0$ implies $|B - B_3| \geqslant 10$. For simplicity we write $B - B_3$ instead of $B - (B_3 \cup B_3 \cup B_3)$. Let Q be a w-set and let D be a collection of

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3-subsets of Q, called blocks, such that each 2-subset of Q is contained in 0 or 3 blocks non repeated. (Q, D) is said a partial $S_3(2, 3, w)$. An element x of Q has degree $d(x) = h$ if x belongs to exactly h blocks of D. The degree-set of (Q, D) is the n-tuple $DS = [d(x), d(y), ...]$, where $x, y, ...$ are the elements of Q. If there are r_i elements of Q having degree h_i, for $i = 1, 2, ..., s$, we write $DS = [(h_i)_{r_i}, (h_2)_{r_2}, ..., (h_s)_{r_s}]$, where $r_1 + r_2 + \cdots + r_s = |Q|$. If $r_i = 1$, for some i, then we write $(h_i)_1 = h_i$.

Obviously $d(x) \equiv 0 \pmod 3$, $6 \leqslant d(x) \leqslant 3(|Q| - 1)/2$ and $|Q| \leqslant |D|/2$. Clearly $(\bigcup_{b \in B - B_3} b, B - B_3)$ is a partial $S_3(2, 3, v)$. In the following we will put $Q = \bigcup_{b \in B - B_3} b$ and $D = B - B_3$.

Our aim is to determine the sets $R(v)$. It is easily seen that $R(3) = \{1\}$ and $R(5) = \{0\}$. Let us first dispose of the case $v \equiv 1$ or $3 \pmod 6$, that is, when there exists a Steiner triple system of order v (STS(v)). Following [2], let $J(v) = \{k/\exists$ a pair of STS(v)'s having exactly k triples in common$\}$. Denote $I_v = \{0, 1, ..., t_v\} - \{t_v - 1, t_v - 2, t_v - 3, t_v - 5\}$, where $t_v = v(v-1)/6$. It was shown in [2] that for $v \geqslant 13$, $J(v) = I_v$, $J(3) = \{1\}$, $J(7) = \{0, 1, 3, 7\}$, and $J(9) = \{0, 1, 2, 3, 4, 6, 12\}$.

Let (S, T_1) and (S, T_2) be two STS(v)'s such that $|T_1 \cap T_2| = k$. Clearly $(S, T_1 \cup T_1 \cup T_2)$ is a (decomposable) $S_3(2, 3, v)$ with k three-times repeated blocks. Hence $J(v) \subseteq R(v)$. Clearly $t_v - 1, t_v - 2, t_v - 3 \notin R(v)$. In order to show that $R(v) = J(v)$ for $v \equiv 1$ or $3 \pmod 6$, we have to show only that there is no indecomposable $S_3(2, 3, v)$ (V, B) such that $|B - B_3| = 15$ and $5, 8 \notin R(9)$.

THEOREM 2.1. *Let* $v \equiv 1$ *or* $3 \pmod 6$. *Then* $R(v) = J(v)$.

Proof. Let (V, B) be a $S_3(2, 3, v)$ with $|B_3| = t_v - 5$. Obviously $|D| = 15$. The degree-set of (Q, D) is $[9, (6)_6]$. Let $Q = \{1, 2, ..., 7\}$ and $d(1) = 9$. If $\{1, 2, 3\} \in D$ is a two-times repeated block, we have, within isomorphism, $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{2, 4, 5\}$, $\{3, 1, 6\}$, $\{3, 5, 6\}$, $\{3, 5, 6\} \in D$. This implies $d(5) > 6$. If $\{1, 2, 3\}$ is not a two-time repeated block we have $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{3, 4, 5\}$, $\{1, 4, 5\} \in D$; this is impossible, hence $t_v - 5 \notin R(v)$.

Let (V, B) be a $S_3(2, 3, 9)$ with $|B_3| = 5$. Hence $|D| = 21$. The degree-set of (Q, D) is either $[(6)_3, (9)_5]$ or $[(6)_6, (9)_3]$ or $[(6)_7, 9, 12]$.

Let $DS = [(6)_3, (9)_5]$. Within isomorphism, it is $B_3 = \{\{9, 1, 2\}$, $\{9, 3, 4\}$, $\{9, 5, 6\}$, $\{9, 7, 8\}$, $\{1, 3, 5\}\}$. It follows that $\{2, 4, 6\} \in D \cap B_3$.

Let $DS = [(6)_6, (9)_3]$ and $d(7) = d(8) = d(9) = 9$. Obviously for every $b \in B_3$ it is $|b \cap \{7, 8, 9\}| \leqslant 1$. Let $\{7, 1, 2\} \in B_3$, then $\{1, 8, 9\} \in D$. If $\{1, 3, 4\} \in B_3$ we have $\{1, 5, 6\} \in D$ and $\{5, 8\}$ or $\{5, 9\}$ is contained in a block $b_1 \in B$ and in a block $b_2 \in D$.

If $DS = [(6)_7, 9, 12]$ let $\{9, 1, 3\} \in B_3$ with $d(9) = 9$. It follows that $|D| > 24$, hence $5 \notin R(9)$.

At last we prove that $8 \notin R(9)$. If $|B_3| = 8$ it is $D = \{\{1, 2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{1, 4, 5\}, \{6, 2, 3\}, \{6, 2, 4\}, \{6, 2, 4\}, \{6, 3, 5\}, \{6, 3, 5\}, \{6, 4, 5\}\}$. This is impossible since there is not a B_3 such that $D \cup B_3 = B$. ∎

3. The Case $v \equiv 5 \pmod 6$

Let (V, B) be a $S_3(2, 3, v)$ with $v \equiv 5 \pmod 6$. Denote $s_v = (|B| - 10)/3 = [v(v-1) - 20]/6$. From $|B - B_3| \geqslant 10$ it follows that $|B_3| \leqslant s_v$. Let $I'_v = \{0, 1, ..., s_v\} - \{s_v - 1, s_v - 2\}$.

Lemma 3.1. *If $k \in R(v)$ then $k + \sigma(v) \in R(2v+1)$, where $\sigma(v) \in \{0, 1, ..., u = v(v+1)/2\} - \{u-1, u-2, u-3, u-5\}$ if $v \geqslant 11$ and $\sigma(5) \in \{0, 1, 2, 3, 5, 6, 7, 9, 15\}$.*

Proof. Let (V, B) be a $S_3(2, 3, v)$ such that $|B_3| = k$ and $V = \{a_1, a_2, ..., a_v\}$. Let W be a $(v+1)$-set such that $|V \cap W| = 0$. Let $F = \{F_1, F_2, ..., F_v\}$ and $G = \{G_1, G_2, ..., G_v\}$ be two one-factorizations of W having $\sigma(v)$ edges in common [3]. Let $C = \{\{a_i, x, y\}/\{x, y\} \in F_i \cup F_i \cup G_i,\ i = 1, 2, ..., v\}$. $(V \cup W, B \cup C)$ is a $S_3(2, 3, 2v+1)$ such that $|(B \cup C)_3| = k + \sigma(v)$. ∎

From the above Lemma it follows easily

Lemma 3.2. *For $v \geqslant 11$, $R(v) = I'_v$ implies $R(2v+1) = I'_{2v+1}$.*

Similarly to Lemmas 6 and 7 of [2], it is possible to prove respectively the following two lemmas.

Lemma 3.3. *Let $v \geqslant 11$. If $k \in R(v)$ then $k + \beta(v+7)/2 + \delta v + \gamma \in R(2v+7)$ for every $\beta = 0, 1, ..., v-2, v$; $\delta = 0, 1$; $\gamma = 0, 1, 3, 7$.*

Lemma 3.4. *For $v \geqslant 11$, $R(v) = I'_v$ implies $R(2v+7) = I'_{2v+7}$.*

Lemma 3.5. *For $v \geqslant 11$, $s_v - 1, s_v - 2 \notin R(v)$.*

Proof. Let (V, B) be a $S_3(2, 3, v)$ such that $|B_3| = s_v - 1$. Hence (Q, D), $Q = \bigcup_{b \in B - B_3} b$, $D = B - B_3$, is a partial $S_3(2, 3, v)$ such that $|D| = 13$ and $d(x) = 6$ for every $x \in Q$. Then we obtain $6x = 39$; this is impossible.

Now let (V, B) be a $S_3(2, 3, v)$ such that $|B_3| = s_v - 2$. Obviously $|D| = 16$ and (Q, D) has degree-set either $[(6)_8]$ or $[(6)_5, (9)_2]$. In the first case let $Q = \{1, 2, ..., 8\}$. Let 1, 2, 3 be the elements such that $\{4, 1\}, \{4, 2\}, \{4, 3\} \subset b \in D$. If $\{1, 2, 3) \notin D$ then $\{1, 2, x\} \in D$ and $\{4, x\} \subset b \in D$. If

TABLE I

D						B'_3			B_3''								
9	10	3	10	11	5	9	6	7	9	6	7	9	8	17	9	2	16
9	10	4	10	4	5	9	2	8	9	14	13	9	12	15	10	6	8
9	10	5	1	2	3	10	6	8	10	7	17	10	1	15	10	13	16
9	1	11	1	2	11	10	7	1	10	14	12	11	6	17	11	7	8
9	1	3	1	2	11	11	6	3	11	3	12	11	15	16	11	13	14
9	1	3	11	4	5	11	7	8	4	6	16	4	7	3	4	8	1
9	11	4				4	6	1	4	17	14	4	2	15	4	12	13
9	11	5				4	7	2	5	6	12	5	7	13	5	8	15
9	4	5				4	8	3	5	17	2	5	1	14	5	3	16
10	2	3				5	6	2	6	1	13	6	2	14	6	3	15
10	2	3				5	7	3	7	1	16	7	2	12	7	14	15
10	2	11				5	8	1	8	2	13	8	3	14	8	12	16
10	11	4							17	1	12	17	3	13	17	16	14

$\{1, 2, 3\} \in D$ then $\{3, x, y\} \in D$ and $\{4, x\}, \{4, y\} \subset b \in D$. It follows that $\{1, x\}$, $\{1, y\} \subset b \in D$.

Let $DS = [(6)_5, (9)_2]$ and $Q = \{1, 2, ..., 7\}$. Then we obtain, within isomorphism, that $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 5\}$, $\{2, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\}$, $\{1, 2, 3\} \in D$. This is impossible. ∎

LEMMA 3.6. *$s_v - 3$, $s_v - 5 \in R(v)$ for every $v \geqslant 11$, $s_v - 4 \in R(v)$ for every $v \geqslant 17$.*

Proof. Let $S' = \{1, 2, ..., 11\}$ and $S'' = \{1, 2, ..., 17\}$. Let D, B'_3, B''_3 be the block-sets of Table 1 (Table II). Then $(S', D \cup B'_3 \cup B'_3 \cup B'_3)$ and $(S'', D \cup B''_3 \cup B''_3 \cup B''_3)$ are a $S_3(2, 3, 11)$ and a $S_3(2, 3, 17)$ with

TABLE II

D						B'_3			B''_3								
8	1	2	8	1	2	7	8	9	7	8	9	7	10	11	7	12	13
8	1	3	8	3	4	7	10	11	7	14	16	7	15	17	4	1	10
8	3	4	8	2	4	4	9	10	4	9	11	4	12	16	4	15	14
9	1	3	9	1	5	4	1	11	4	17	13	6	1	13	6	8	14
9	1	5	9	3	6	3	2	10	6	10	12	6	11	15	6	17	16
9	3	6	9	5	6	3	5	11	1	11	12	1	15	16	1	14	17
7	4	5	7	4	6	1	6	10	5	3	17	5	8	15	5	10	13
7	4	3	7	2	6	2	9	11	5	11	16	5	12	14	3	2	12
7	2	5	7	2	1	5	8	10	2	9	16	2	10	14	2	11	17
7	5	1	7	6	3	6	8	11	2	13	15	3	10	15	3	11	14
7	1	3	4	2	6				3	13	16	8	11	13	8	12	17
4	2	5	4	5	6				9	10	17	9	14	13	9	12	15
2	5	6							8	10	16						

TABLE III

D						B_3											
1	2	3	1	2	9	1	5	11	1	7	13	1	4	15	3	8	17
1	2	10	1	3	9	3	11	12	4	12	13	9	10	15	9	13	14
1	3	10	1	9	10	1	6	12	1	8	14	1	16	17	4	5	14
2	3	9	2	3	10	4	6	16	4	7	17	10	11	14	10	13	17
2	9	10	3	9	10	10	16	12	2	5	12	2	6	13	2	7	14
10	4	5	10	4	5	4	8	11	5	9	16	5	15	17	14	15	16
10	4	6	10	7	6	11	13	16	12	14	17	2	11	17	2	8	16
10	7	6	10	7	5	2	3	15	6	9	17	6	15	11	7	9	11
4	6	8	4	6	8	3	5	13	3	6	14	3	7	16	7	15	12
4	5	8	8	5	7	8	9	12	8	15	13						
8	5	7	8	6	7												

$|B_3'| = s_{11} - 3$ and $|B_3''| = s_{17} - 3$ $[|B_3'| = s_{11} - 5$ and $|B_3''| = s_{17} - 5]$ three-times repeated blocks, respectively.

Let D and B_3 be the block-sets of Table III, and $(S'', D \cup B_3 \cup B_3 \cup B_3)$ is a $S_3(2, 3, 17)$ with $|B_3| = s_{17} - 4$.

Lemmas 3.1, 3.2, 3.3, and 3.4 complete the proof. ∎

THEOREM 3.1. $R(11) = I_{11}' - \{11\}$.

Proof. Let (V', B') be the (only) $S_3(2, 3, 5)$ and let $V_1 = \{a_1, a_2, ..., a_5\}$. It is easy to construct three one-factorizations $F = \{F_1, F_2, ..., F_5\}$, $G = \{G_1, G_2, ..., G_5\}$, and $H = \{H_1, H_2, ..., H_5\}$ of $V'' = \{1, 2, ..., 6\}$, such that $|\{\{x, y\}/\{x, y\} \in F_i \cap G_i \cap H_i$ for some $i = 1, 2, ..., 5\}| = 4$.

Let $V = V' \cup V''$ and let $B'' = \{\{a_i, x, y\}/\{x, y\} \in F_i \cup G_i \cup H_i$, $i = 1, 2, ..., 5\}$. Then $(V, B' \cup B'')$ is a $S_3(2, 3, 11)$ with 4 three-times repeated blocks.

Let D and B_3 be the block-sets of Table IV. Let $V = \{1, 2, ..., 11\}$, then $(V, D \cup B_3 \cup B_3 \cup B_3)$is a $S_3(2, 3, 11)$ with $|B_3| = 8$.

TABLE IV

D												B_3					
1	6	4	1	6	4	1	6	5	1	7	4	11	1	9	11	2	4
1	7	5	1	7	5	2	8	5	2	8	7	11	3	5	11	10	7
2	8	7	2	6	5	2	6	5	2	6	7	11	8	6	1	2	10
3	10	4	3	10	6	3	10	6	3	7	6	1	3	8	2	3	9
3	7	4	3	7	4	9	10	8	9	10	6						
9	10	5	9	8	7	9	8	4	9	6	4						
9	7	4	9	4	5	9	5	7	4	5	10						
4	5	8	4	10	8	5	10	8									

Now we prove that $11 \notin R(11)$. Let (V, B) be a $S_3(2, 3, 11)$ such that $|B_3| = 11$ and $V = \{1, 2, ..., 11\}$. Hence (Q, D), $Q = \bigcup_{b \in B - B_3} b$, $D = B - B_3$, is a partial $S_3(2, 3, 11)$ such that $|D| = 22$. (Q, D) may have one of the following degree-sets: $[(6)_7, (12)_2]$, $[(6)_5, (9)_4]$, $[(6)_6, (9)_2, 12]$, $[(6)_8, (9)_2]$, $[(6)_9, 12]$, $[(6)_{11}]$.

Let $DS[(6)_7, (12)_2]$. Let $1, 2 \in Q$ be such that $d(1) = d(2) = 12$ and let $10, 11 \in V - Q$. Then $\{1, 10, 11\}$, $\{2, 10, 11\} \in B_3$; this is impossible.

Let $DS = [(6)_5, (9)_4]$. Let $d(1) = d(2) = \cdots = d(5) = 6$, $d(6) = d(7) = \cdots = d(9) = 9$, and $V - Q = \{10, 11\}$. It is either $\{1, 10, 11\} \in B_3$ or $\{6, 10, 11\} \in B_3$. In the first case it follows $\{1, 2, 3\}$, $\{1, 4, 5\} \in B_3$, hence $\{\alpha, \beta, \gamma\}$, $\{\alpha, \beta, \delta\}$, $\{\alpha, \gamma, \delta\}$, $\{\beta, \gamma, \delta\} \in D$ and $|\{\alpha, \beta, \gamma, \delta\} \cap \{1, 2, 3\}| = 0$. Moreover it is $\{\alpha, \beta, \gamma, \delta\} = \{6, 7, 8, 9\}$ and $\{1, 6, 7\}$, $\{1, 6, 8\}$, $\{1, 6, 9\}$, $\{1, 7, 9\}$, $\{1, 7, 9\}$, $\{1, 8, 9\} \in D$. This is impossible.

If $\{6, 10, 11\} \in B_3$ we have $\{6, 4, 5\}$, $\{1, 2, 3\} \in B_3$. Let $\{\alpha, \beta, \gamma\}$ be the block of D such that $|\{\alpha, \beta, \gamma\} \cap \{6, 4, 5\}| = 0$. It follows that $d(\alpha) = 7$ or 10.

Let $DS = [(6)_6, (9)_2, 12]$. Let $d(1) = d(2) = \cdots = d(6) = 6$, $d(7) = d(8) = 9$, $d(9) = 12$, and $V - Q = \{10, 11\}$. There is $b \in D$ such that $\{7, 8\} \subset b$. Let $A(x)$ be the set of $y \in Q$ such that $\{x, y\} \subset b$ for some $b \in D$. It is, within isomorphism, $A(7) = \{1, 2, 3, 4, 8, 9\}$ and $A(8) = \{1, 2, 5, 6, 7, 9\}$.

Case 1. There is a $b \in D$ such that $\{1, 2\} \subset b$. If $\{1, 2, \beta\}$ is a two-times repeated block of D we have $\{1, 2, \beta\}$, $\{1, 2, \beta\}$, $\{1, 2, \gamma\}$, $\{1, \alpha, \beta\}$, $\{1, \alpha, \gamma\}$, $\{1, \alpha, \gamma\}$, $\{2, \alpha, \beta\} \in D$ with $\{\alpha, \beta, \gamma\} = \{7, 8, 9\}$; this is impossible. Hence we have $\{1, 2, 7\}$, $\{1, 2, 8\}$, $\{1, 2, 9\}$, $\{1, 7, 8\}$, $\{1, 7, 9\}$, $\{1, 8, 9\}$, $\{2, 7, 8\}$, $\{2, 7, 9\}$, $\{2, 8, 9\}$, $\{7, 8, 9\} \in D$. This implies $\{7, 3, 4\} \in D \cap B_3$.

Case 2. There is not a $b \in D$ such that $\{1, 2\} \subset b$. We have $\{1, 3, \alpha\}$, $\{1, 3, \alpha\}$, $\{1, 3, \beta\}$, $\{1, 8, \beta\}$, $\{1, 8, \beta\}$, $\{1, 8, \alpha\} \in D$ with $\{\alpha, \beta\} = \{7, 9\}$; hence $\{3, \gamma, \beta\}$, $\{3, \gamma, \beta\}$, $\{3, \gamma, \alpha\} \in D$. Since $3 \notin A(8)$ and $5, 6 \notin A(7)$ we have $\gamma \neq 5, 6, 8$. Hence $\gamma = 4$ and $\{4, \delta, \beta\}$, $\{4, \delta, \alpha\}$, $\{4, \delta, \alpha\} \in D$. From $5, 6 \notin A(7)$, $4 \notin A(8)$, and $d(1) = d(3) = 6$ we obtain $\delta \neq 5, 6, 8, 1, 3$. Then $\delta = 2$ and $\{2, 8, \beta\}$, $\{2, \gamma, \beta\}$, $\{2, 8, \alpha\} \in D$. This implies $\{8, 5, 6\} \in D \cap B_3$.

Before proceeding, we observe that if $d(1) = 6$ and $d(x) = 6$ for every $x \in A(1)$, we have either $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\} \in D$ or $\{1, 2, 3\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{1, 4, 5\}$, $\{2, 4, x\}$, $\{2, 4, x\}$, $\{2, 3, x\}$, $\{3, 5, x\}$, $\{3, 5, x\}$, $\{4, 5, x\} \in D$ with $d(x) = 6$ or 12.

Let $DS = [(6)_8, (9)_2]$. For every $\alpha \in Q$ let $Y(\alpha)$ be the set of $\beta \in A(\alpha)$ such that $d(\beta) = 9$. Obviously $|Y(\alpha)| \geqslant 1$ for every α of degree 6. It is easy to see that there is an α such that $d(\alpha) = 6$ and $|Y(\alpha)| = 1$. Pose $\alpha = 1$ and $Y(1) = \{6\}$. Then we have either $\{1, 6, 2\}$, $\{1, 6, 2\}$, $\{1, 6, 3\}$, $\{1, 2, 4\}$,

$\{1, 3, 4\}$, $\{1, 3, 4\}$, $\{2, 4, 5\}$, $\{2, 4, 5\}$, $\{2, 6, 5\}$, $\{3, 4, 5\}$, $\{3, 5, 6\}$, $\{3, 5, 6\} \in D$ or $\{1, 6, 2\}$, $\{1, 6, 3\}$, $\{1, 6, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 6\}$, $\{2, 4, 6\}$, $\{2, 3, 4\}$, $\{3, 4, 6\} \in D$ hence $|D \cap B_3| > 1$.

Let $DS = [(6)_9, 12]$ and let $d(1) = d(2) = \cdots = d(9) = 6$, $d(10) = 12$. Let 1 be an element of Q such that $d(x) = 6$ for every $x \in A(1)$. It follows, within isomorphism, that

$$\begin{aligned} D = \{&\{1,2,3\}, \{1,2,3\}, \{1,2,4\}, \{1,3,5\}, \{1,4,5\}, \{1,4,5\},\\ &\{2,4,10\}, \{2,4,10\}, \{2,3,10\}, \{3,5,10\}, \{3,5,10\},\\ &\{4,5,10\}, \{6,7,10\}, \{6,8,10\}, \{6,9,10\}, \{7,8,10\},\\ &\{7,9,10\}, \{8,9,10\}, \{6,7,8\}, \{6,7,9\}, \{6,8,9\}, \{7,8,9\}\}. \end{aligned}$$

It is easy to see that (Q, D) is not embeddable in a $S_3(2, 3, 11)$.

Similarly we obtain the proof if $DS[(6)_{11}]$. ▌

THEOREM 3.2. $R(17) = I'(17)$.

Proof. Lemmas 3.5 and 3.6 imply 40, 41, 37, 38, $39 \in R(17)$. Let $V = \{1, 2, ..., 17\}$. Let D_1, D_2, B'_3, B''_3, B'''_3 be the block-sets of Table V and let D be the block-set given in Table I. Let $\bar{D} = D \cup D_1$, $B_3 = B''_3 \cup B'''_3$, $\bar{D}_1 = D \cup D_2$, $\bar{B}_3 = B'_3 \cup B'''_3$, and $\bar{D}_2 = D \cup D_1 \cup D_2$. Then $(V, \bar{D} \cup B_3 \cup B_3 \cup B_3)$, $(V, \bar{D}_1 \cup \bar{B}_3 \cup \bar{B}_3 \cup \bar{B}_3)$, and $(V, \bar{D}_2 \cup B''' \cup B''' \cup B''')$ are $S_3(2, 3, 17)$ with 35, 33, and 29 three-times repeated blocks, respectively.

TABLE V

D_1			D_2			B'_3			B''_3			B'''_3					
9	6	7	4	6	16	9	6	7	4	6	16	9	2	16	9	14	13
9	6	8	4	7	3	9	8	17	4	7	3	9	12	15	10	1	15
9	6	8	4	12	13	10	6	8	4	12	13	10	13	16	10	14	12
9	8	17	4	6	16	10	7	17	5	6	12	11	6	17	11	7	8
9	7	17	4	7	3				5	7	13	11	3	12	11	15	16
9	7	17	4	12	13				5	3	16	11	13	14	4	8	1
10	6	8	4	3	16							4	17	14	4	2	15
10	6	7	4	6	12							5	8	15	5	17	2
10	6	7	4	7	13							5	1	14	6	1	13
10	8	17	5	6	16							6	2	14	6	3	15
10	8	17	5	7	3							7	1	16	7	2	12
10	7	17	5	12	13							7	14	15	8	2	13
			5	3	16							8	3	14	8	12	16
			5	6	12							17	1	12	17	3	13
			5	7	13							17	16	14			
			5	3	16												
			5	6	12												
			5	7	13												

Let T, D_1 be the block-sets of Table VI. Let $M=\{\{i, x, y\}/\{x, y\}\in H_i, i=1, 2, ..., 5\}$ and $B=T\cup D_1\cup M\cup M\cup M$. Then (V, B) is a $S_3(2, 3, 17)$ having 42 three-times repeated blocks.

Replacing in B all the blocks $\{1, 6, 7\}$, $\{1, 8, 9\}$, $\{2, 6, 8\}$, $\{2, 7, 9\}$, $\{4, 6, 10\}$, $\{4, 7, 13\}$, $\{4, 14, 15\}$, $\{5, 6, 14\}$, $\{5, 7, 15\}$, $\{5, 3, 10\}$ with the block-set T_1 given in Table VI we obtain $32\in R(17)$.

Let L be the set of all the unordered pairs on $\{6, 7, 8, 9\}$. Replacing in B all the blocks $\{1, 6, 7\}$, $\{1, 8, 9\}$, $\{2, 6, 8\}$, $\{2, 7, 9\}$, $\{3, 6, 9\}$, $\{3, 7, 8\}\in M$ with the block-set $\{\{i, x, y\}/\{x, y\}\in L, i=1, 2, 3\}$, we obtain $36\in R(17)$.

Let $\bar{B}$ be the block-set obtained from B by replacing all the blocks $\{1, 11, 13\}$, $\{1, 16, 15\}$, $\{1, 14, 17\}$, $\{1, 12, 10\}$, $\{2, 12, 13\}$, $\{2, 11, 17\}$, $\{2, 15, 10\}$, $\{2, 16, 14\}\in M$ with the block-set T_2 of Table VI. Then $(V, \bar{B})$ is a $S_3(2, 3, 17)$ with 34 three-times repeated blocks.

Similarly we obtain $26\in R(17)$ by replacing in $\bar{B}$ the blocks $\{3, 11, 12\}$, $\{3, 13, 14\}$, $\{3, 17, 10\}$, $\{3, 15, 16\}$, $\{1, 6, 7\}$, $\{1, 8, 9\}$, $\{2, 6, 8\}$, $\{2, 7, 9\}$ with the union of the blocks given in D_1 of Table V with the following

TABLE VI

T			D_1			T_1						T_2					
1	2	3	6	11	15	1	6	7	4	7	3	1	11	13	2	16	14
1	2	4	6	12	16	1	6	7	4	6	14	1	11	13	2	16	14
1	2	5	6	13	17	1	8	9	4	7	15	1	16	15	2	11	13
1	3	4	7	11	10	1	6	8	4	3	10	1	16	15	2	16	15
1	3	5	7	12	14	1	8	9	5	6	10	1	14	17	2	14	17
2	3	4	7	16	17	1	7	9	5	14	15	1	14	17	2	10	12
2	3	5	8	12	15	2	6	7	5	7	3	1	12	10	2	15	10
2	4	5	8	13	16	2	6	8	5	6	14	1	12	10	2	15	10
3	4	5	8	14	10	2	6	8	5	7	15	1	12	13	2	12	13
1	4	5	9	11	14	2	7	9	5	3	10	1	11	17			
			9	13	15	2	7	9	5	6	14	1	15	10			
			9	10	16	2	8	9	5	7	15	1	16	14			
						4	6	10	5	3	10	2	12	13			
						4	14	15	4	14	15	2	11	17			
						4	7	3	4	6	10	2	11	17			

H_1		H_2		H_3		H_4		H_5	
11	13	12	13	11	12	6	10	6	14
6	7	6	8	6	9	7	13	7	15
8	9	7	9	7	8	8	11	8	17
16	15	11	17	13	14	9	16	9	12
14	17	15	10	17	10	12	17	11	16
12	10	16	14	15	16	14	15	13	10

blocks: $\{i, 11, 15\}$, $\{i, 16, 15\}$, $\{i, 14, 17\}$, $\{i, 2, 10\}$, $\{i, 12, 13\}$, $\{i, 11, 17\}$, $\{i, 15, 10\}$, $\{i, 16, 14\}$, $\{i, 11, 12\}$, $\{i, 13, 14\}$, $\{i, 17, 10\}$, $\{i, 15, 16\}$ for every $i = 1, 2, 3$.

Let B^* be the block-set obtained from B by replacing all the blocks b such that $j \in b$ for every $j = 1, 2, ..., 5$ with the following block-set: $\{\{i, x, y\}/\{x, y\} \in H_1 \cup H_2 \cup H_3,\ i = 1, 2, 3\} \cup \{\{4, x, y\}/\{x, y\} \in H_4 \cup H_4 \cup H_5\} \cup \{\{5, x, y\}/\{x, y\} \in H_4 \cup H_5 \cup H_5\}$. Then (V, B^*) is a $S_3(2, 3, 17)$ with 22 three-times repeated blocks.

Let D and B_3'' be the block-sets of Table II. Let

$$L_1 = \{\{12, 13\}, \{14, 16\}, \{15, 17\}\},$$

$$L_2 = \{\{12, 16\}, \{15, 14\}, \{17, 13\}\},$$

$$M_1 = \{\{7, x, y\}/\{x, y\} \in L_1\} \cup \{\{4, x, y\}/\{x, y\} \in L_2\} \subseteq B_3'',$$

$$M_2 = \{\{7, x, y\}/\{x, y\} \in L_1 \cup L_1 \cup L_2\}$$
$$\cup \{\{4, x, y\}/\{x, y\} \in L_1 \cup L_2 \cup L_2\},$$

$\bar{B}_3 = B_3'' - M_1$ and $\bar{D} = D \cup M_2$. Then $(V, \bar{D} \cup \bar{B}_3 \cup \bar{B}_3 \cup \bar{B}_3)$ is a $S_3(2, 3, 17)$ with $|\bar{B}_3| = 31$.

Let (W, C) be the (only) $S_3(2, 3, 5)$ and $W = \{13, 14, 15, 16, 17\}$. Let P_1, P_2, F_1, F_2, ..., F_5, G_1, G_2, ..., G_{10} be the block-sets of Table VII. Let K_{12} be

TABLE VII

P_1			P_2			F_1		F_2		F_3		F_4		F_5	
1	2	4	1	3	4	1	5	1	6	1	7	1	8	1	9
2	3	5	2	4	5	2	8	2	10	2	9	2	7	2	6
3	4	6	3	5	6	3	9	3	8	3	10	3	11	3	7
4	5	7	4	6	7	4	10	4	9	4	8	4	12	4	11
5	6	8	5	7	8	11	6	11	7	11	5	5	9	8	12
6	7	9	6	8	9	12	7	12	5	12	6	6	10	5	10
7	8	10	7	9	10										
8	9	11	8	10	11										
9	10	12	9	11	12										
10	11	1	10	12	1										
11	12	2	11	1	2										
12	1	3	12	2	3										

G_1		G_2		G_3		G_4		G_5		G_6		G_7		G_8		G_9		G_{10}	
1	5	1	6	1	7	1	6	1	7	1	5	1	7	1	6	1	5	1	6
2	10	2	8	2	9	2	9	2	8	2	8	2	10	2	8	11	6	2	10
3	8	3	9	3	10	3	10	3	9	3	9	3	8	3	9	12	7	3	8
4	9	4	10	4	8	4	8	4	10	4	10	4	9	4	10	2	10	4	9
7	11	11	5	11	6	11	5	11	6	11	7	11	5	11	7	3	8	11	5
6	12	12	7	12	5	12	7	12	5	12	6	12	6	12	5	4	9	12	7

the complete graph on the vertex-set $\{1, 2, ..., 12\}$. K_{12} may be factored [4] in the set of triangles P_1 (or P_2) and the set of one-factors $F_1, F_2, ..., F_5$.

Let

$$\begin{aligned} D_1 &= \{\{13, x, y\}/\{x, y\} \in G_6 \cup F_1 \cup F_1\} \\ &\cup \{\{14, x, y\}/\{x, y\} \in F_2 \cup F_2 \cup F_3\} \\ &\cup \{\{15, x, y\}/\{x, y\} \in G_{10} \cup G_3 \cup F_3\}, \\ D_2 &= \{\{16, x, y\}/\{x, y\} \in F_4 \cup F_4 \cup F_5\} \\ &\cup \{\{17, x, y\}/\{x, y\} \in F_4 \cup F_5 \cup F_5\}, \\ D_3 &= \{\{16, x, y\}/\{x, y\} \in F_4 \cup F_4 \cup F_4\} \\ &\cup \{\{17, x, y\}/\{x, y\} \in F_5 \cup F_5 \cup F_5\}. \end{aligned}$$

Then $(V, P_1 \cup P_1 \cup P_2 \cup D_1 \cup D_2)$, $(V, P_1 \cup P_1 \cup P_2 \cup D_1 \cup D_3)$, $(V, P_1 \cup P_1 \cup P_1 \cup D_1 \cup D_3)$ are three $S_3(2, 3, 17)$ with 4, 16, and 28 three-times repeated blocks, respectively.

Let

$$\begin{aligned} D &= \{\{13, x, y\}/\{x, y\} \in G_6 \cup F_1 \cup F_1\} \\ &\cup \{\{14, x, y\}/\{x, y\} \in F_2 \cup F_2 \cup F_5\} \\ &\cup \{\{15, x, y\}/\{x, y\} \in G_{10} \cup G_3 \cup F_3\} \\ &\cup \{\{16, x, y\}/\{x, y\} \in F_4 \cup F_4 \cup F_4\} \\ &\cup \{\{17, x, y\}/\{x, y\} \in F_5 \cup F_5 \cup F_3\}. \end{aligned}$$

Then $(V, P_1 \cup P_1 \cup P_2 \cup D)$ is a $S_3(2, 3, 17)$ with 10 three-times repeated blocks.

For every $v \in \{0, 1, 2, 3, 5\}$ let α_v be the permutation of $\{1, 2, ..., 5\}$ fixing exactly v elements. Let $D_v^* = \{\{i+12, x, y\}/\{x, y\} \in F_i \cup F_i \cup F_{i\alpha_v}, i = 1, 2, ..., 5\}$. Then for every v, $(V, P_1 \cup P_1 \cup P_2 \cup D_v^*)$ and $(V, P_1 \cup P_1 \cup P_1 \cup D_v^*)$ are two $S_3(2, 3, 17)$ with v and $v + 12$ three-times repeated blocks. Hence $0, 6, 12, 18, 24, 30, 42 \in R(17)$.

Let

$$\begin{aligned} X_1 &= \{\{13, x, y\}/\{x, y\} \in F_1 \cup F_1 \cup G_1\} \\ &\cup \{\{14, x, y\}/\{x, y\} \in F_2 \cup F_3 \cup G_2\}, \\ X_2 &= \{\{15, x, y\}/\{x, y\} \in F_2 \cup F_3 \cup G_3\} \\ &\cup \{\{16, x, y\}/\{x, y\} \in F_4 \cup F_4 \cup F_5\} \\ &\cup \{\{17, x, y\}/\{x, y\} \in F_4 \cup F_5 \cup F_5\}, \end{aligned}$$

$$X_3 = \{\{15, x, y\}/\{x, y\} \in F_4 \cup F_3 \cup G_3\}$$
$$\cup \{\{16, x, y\}/\{x, y\} \in F_2 \cup F_4 \cup F_4\}$$
$$\cup \{\{17, x, y\}/\{x, y\} \in F_5 \cup F_5 \cup F_5\},$$
$$X_4 = \{\{15, x, y\}/\{x, y\} \in F_2 \cup F_3 \cup G_3\}$$
$$\cup \{\{16, x, y\}/\{x, y\} \in F_4 \cup F_4 \cup F_4\}$$
$$\cup \{\{17, x, y\}/\{x, y\} \in F_5 \cup F_5 \cup F_5\}.$$

Then $(V, P_1 \cup P_1 \cup P_2 \cup X_1 \cup X_2)$, $(V, P_1 \cup P_1 \cup P_2 \cup X_1 \cup X_3)$, $(V, P_1 \cup P_1 \cup P_2 \cup X_1 \cup X_4)$, $(V, P_1 \cup P_1 \cup P_1 \cup X_1 \cup X_3)$, and $(V, P_1 \cup P_1 \cup P_1 \cup X_1 \cup X_4)$ are $S_3(2, 3, 17)$ with 1, 7, 13, 19, and 25 three-times repeated blocks, respectively.

Let

$$D_1^* = \{\{13, x, y\}/\{x, y\} \in F_1 \cup F_1 \cup G_1\}$$
$$\cup \{\{14, x, y\}/\{x, y\} \in F_2 \cup F_2 \cup G_4\},$$
$$D_2^* = \{\{15, x, y\}/\{x, y\} \in F_3 \cup F_3 \cup G_5\},$$
$$D_3^* = \{\{16, x, y\}/\{x, y\} \in F_4 \cup F_4 \cup F_5\},$$
$$D_4^* = \{\{17, x, y\}/\{x, y\} \in F_4 \cup F_5 \cup F_5\},$$
$$D_5^* = \{\{16, x, y\}/\{x, y\} \in F_4 \cup F_4 \cup F_4\},$$
$$D_6^* = \{\{17, x, y\}/\{x, y\} \in F_5 \cup F_5 \cup F_5\},$$
$$D_7^* = \{\{13, x, y\}/\{x, y\} \in F_3 \cup F_4 \cup G_5\},$$
$$D_8^* = \{\{16, x, y\}/\{x, y\} \in F_3 \cup F_4 \cup F_5\},$$
$$D_9^* = \{\{16, x, y\}/\{x, y\} \in F_3 \cup F_4 \cup F_4\}.$$

Then $(V, P_1 \cup P_1 \cup P_2 \cup D_1^* \cup D_2^* \cup D_3^* \cup D_4^*)$, $(V, P_1 \cup P_1 \cup P_2 \cup D_1^* \cup D_2^* \cup D_5^* \cup D_6^*)$, $(V, P_1 \cup P_1 \cup P_1 \cup D_1^* \cup D_2^* \cup D_5^* \cup D_6^*)$, $(V, P_1 \cup P_1 \cup P_2 \cup D_1^* \cup D_7^* \cup D_8^* \cup D_4^*)$, $(V, P_1 \cup P_1 \cup P_1 \cup D_1^* \cup D_7^* \cup D_8^* \cup D_4^*)$, $(V, P_1 \cup P_1 \cup P_2 \cup D_1^* \cup D_7^* \cup D_9^* \cup D_6^*)$, $(V, P_1 \cup P_1 \cup P_1 \cup D_1^* \cup D_7^* \cup D_9^* \cup D_6^*)$ are $S_3(2, 3, 17)$ with 3, 15, 27, 2, 14, 8, and 20 three-times repeated blocks, respectively.

Let

$$D_1^{**} = \{\{13, x, y\}/\{x, y\} \in F_1 \cup F_1 \cup G_6\}$$
$$\cup \{\{14, x, y\}/\{x, y\} \in F_2 \cup F_2 \cup G_4\}$$
$$\cup \{\{15, x, y\}/\{x, y\} \in F_3 \cup F_3 \cup F_4\},$$

$$D_2^{**} = \{\{16, x, y\}/\{x, y\} \in F_4 \cup F_4 \cup G_7\},$$

$$D_3^{**} = \{\{17, x, y\}/\{x, y\} \in F_5 \cup F_5 \cup F_5\},$$

$$D_4^{**} = \{\{16, x, y\}/\{x, y\} \in F_4 \cup F_5 \cup G_7\},$$

$$D_5^{**} = \{\{17, x, y\}/\{x, y\} \in F_4 \cup F_5 \cup F_5\}.$$

Then $(V, P_1 \cup P_1 \cup P_2 \cup D_1^{**} \cup D_2^{**} \cup D_3^{**})$, $(V, P_1 \cup P_1 \cup P_1 \cup D_1^{**} \cup D_2^{**} \cup D_3^{**})$, $(V, P_1 \cup P_1 \cup P_2 \cup D_1^{**} \cup D_4^{**} \cup D_5^{**})$, $(V, P_1 \cup P_1 \cup P_1 \cup D_1^{**} \cup D_4^{**} \cup D_5^{**})$ are $S_3(2, 3, 17)$ with 11, 23, 5, and 17 three-times repeated blocks, respectively.

Let

$$\begin{aligned} D = {} & \{\{13, x, y\}/\{x, y\} \in F_1 \cup F_1 \cup G_8\} \\ & \cup \{\{14, x, y\}/\{x, y\} \in F_2 \cup F_3 \cup G_9\} \\ & \cup \{\{15, x, y\}/\{x, y\} \in F_2 \cup F_3 \cup F_4\} \\ & \cup \{\{16, x, y\}/\{x, y\} \in F_3 \cup F_4 \cup F_4\} \\ & \cup \{\{17, x, y\}/\{x, y\} \in F_5 \cup F_5 \cup F_5\}. \end{aligned}$$

Then $(V, P_1 \cup P_1 \cup P_2 \cup D)$, $(V, P_1 \cup P_1 \cup P_1 \cup D)$ are two $S_3(2, 3, 17)$ with 9 and 21 three-times repeated blocks, respectively. ∎

The above Theorems 3.1 and 3.2 and Lemmas 3.1, 3.2, 3.3, and 3.4 imply our main result.

THEOREM 3.3. *$R(v) = I_v'$ for every $v \geqslant 17$. $R(11) = I_{11}' - \{11\}$, $R(5) = \{0\}$.*

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